$$VU^2 - VW^2 = r^2 + r^2 - 2r^2 \cos \angle VOU - (r^2 + r^2 - 2r^2 \cos \angle VOW)$$
$$VU^2 - VW^2 = 2r^2(\cos \angle VOW - \cos \angle VOU)$$

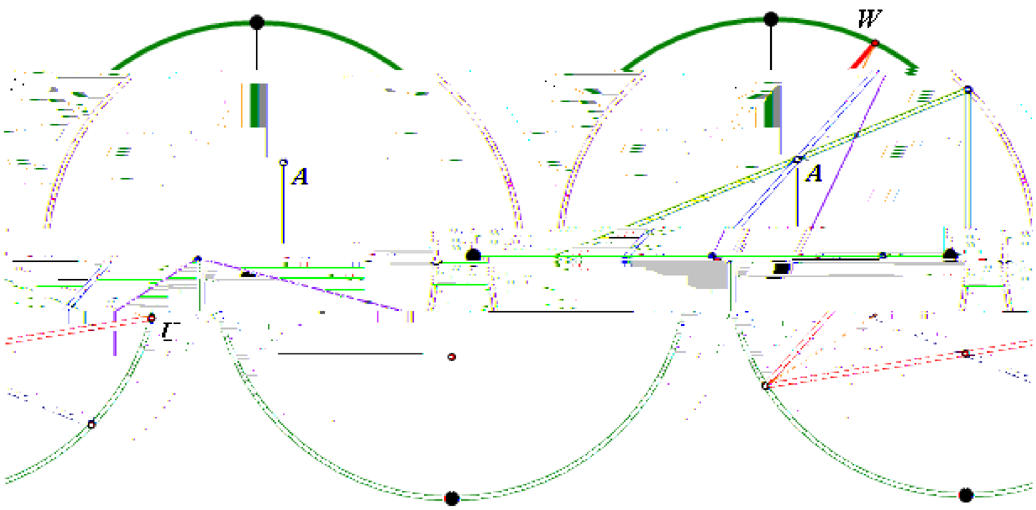

Figure 1. Two solutions from this A and B.

Since cosine is a strictly decreasing function on the interval of our angle sizes, the difference $\cos \angle VOW - \cos \angle VOU = 0$ only at the points emphasized with large dots in Figure 1. Only the two points on the perpendicular bisector of AB give non-degenerate solutions. We include such a simple case because its corresponding hyperbolic case will illustrate our bijection between Euclidean and hyperbolic cases later.

1.0.1 Alhazen's Billiard Problem in Hyperbolic Geometry

Hyperbolic geometry has several models; we will use the Poincaré disk and the Klein model. The Poincaré disk is a hyperbolic space made of the Euclidean points inside, not including, a fixed boundary circle. Euclidean diameters and arcs of circles orthogonal to the boundary circle are hyperbolic lines. Hyperbolic distance cannot be discerned from appearances in the model because the hyperbolic distance formula gives different sizes for segments which look the same in Euclidean size, depending on their proximity to the boundary. One consequence of the hyperbolic distance formula is that a circle in the disk has a Euclidean center and a hyperbolic center which are the same only when the circle's Euclidean center coincides with the center of the boundary circle, called O. Another property which surprises people in their first visit to the Poincaré disk is that the angle sums of triangles lie between 0 and π. The Klein model also uses a disk for the boundary. The hyperbolic lines are Euclidean chords. In this model, Euclidean appearances are even more deceiving because hyperbolic angle sizes are not the Euclidean sizes visible in the model. The Poincaré disk has its angle size visible, using tangents at the vertex of the angle. Both models allow infinitely many intersecting hyperbolic lines parallel to a given line. There exists an isomorphism between these two models, called stereographic projection, which, by coincidence, fits our Alhazen problem.

The Alhazen billiard problem in the hyperbolic disk model has more given than the Euclidean version because we have the disk, with the given circle inside the disk. The given circle has a Euclidean center E, the one we would use to draw the circle with a compass. The given circle also has a hyperbolic center H, the point which is hyperbolic equidistant from the points on the circle. When E is not O, then H ends up closer to the boundary than E, on the Euclidean ray $\overrightarrow{OE}$: (We can construct H by constructing a hyperbolic line perpendicular to both the boundary and the given circle.) Figure 2 summarizes the properties of the hyperbolic situation. For points A and B conveniently placed, the construction is possible. (Our bijection will define what we mean by "convenient.") As a quick look at the hyperbolic version, the given points have

2 Non-constructibility

Theorem There exists a bijection between constructible Euclidean Alhazen triangles and constructible hyperbolic Alhazen triangles, as well as a bijection between the non-constructible cases.

Proof. Suppose we can construct any of the hyperbolic isosceles triangles whose legs contain A and B. Construction of the translation of the given circle to the center O

Figures 1 and 4 illustrate a detail from Alhazen's history. Dörrie's concise presentation [2] gives an analytic proof for there being as many as four Alhazen triangles possible, depending on the placement of the given points A and B. Our bijection said there would be exactly two constructible hyperbolic triangles from this special position of A and B because the Euclidean case had two. We constructed the hyperbolic triangle (the second triangle uses the vertex on the triangle. If the Alhazen given information allows a construction in one geometry, there is a corresponding constructible case in the other geometry. Likewise for the non-constructible cases.

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3.1 References

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