Fibonacci numbers when counting chord diagrams

Jane Kraemer and Michael McDaniel
May 20, 2010

1. Introduction.

diagrams are one-piece and plenty of diagrams with the closed vertex path do not have our restricted structure. We will call any diagram with an isolated chord a null chord diagram for a reason given in the last section. We will call a place where two chords cross a transposition because if $_2$ crosses $_3$, an endpoint of $_3$ appears between the endpoints of $_2$.

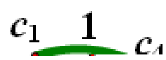

Figure 1. One-piece diagram.

3. Counting null chord diagrams. Let's consider the set of all one-piece diagrams with distinct chords. We are treating chords as distinctly numbered. Because we either have two consecutive chords crossing or not, and there are places where endpoints are neighbors, the size of is 2 . We will now count the number of null chord diagrams in .

We isolate some chord . We will count the diagrams such that no chord preceding is isolated, and every chord after can be isolated or not be isolated. To isolate chord , we cannot transpose endpoints at or endpoints at + 1. However, we must transpose endpoints at 1 in order to keep chord $_1$ not isolated.

To keep the chords before not isolated, some of their endpoints must be transposed, but not necessarily all of them. We can name the arcs where transpositions could happen with the phrase 1 or 2 or both, and, 2 or 3 or both, and,..., and 3 or 2 or both. (We have to handle = separately, after the proof.) We will call the set of all optional choices which leave no chord before isolated the set Each element of is a subset of f1 2 g because each is a set of subsets of transpositions The number is in one of these subsets if and only if the endpoints of $_1$ and are transposed, (except for = 1, when $_1$ and have endpoints transposed.) We will use the notation j j for the number of elements in . This structure will remind number theory fans of a famous sequence.

Lemma. j j = , for where is the Fibonacci number.

Proof. To isolate $_1$, the set of transpositions $_1$ must not include endpoints at arcs 1 or 2. Endpoints beyond $_1$ are not included in $_1$. Thus, we have no optional endpoints to include in $_1$, so the set of chord diagrams with optionally transposed endpoints $_1$ = f ;g and j $_1$j = 1 This leaves 2 endpoints remaining above $_1$, so there are 2 ^{2} ways to isolate chord 1. (We might as well count our chord diagrams while we work the proof.)

To isolate $_2$ with $_1$ not isolated, the set of transpositions $_2$ must not include endpoints at arcs 2 or 3, but must transpose endpoints at 1. So, again there are no options to put in $_2$ because the transposition at 1 is mandatory. So the set $_2$ = f ;g and j $_2$j = 1 (We have just begun our sequence of Fibonacci numbers with 1, 1.) There are 2 ^{3} ways to isolate $_2$ without isolating

The cases through $_1$ all have a power of 2 times a corresponding Fibonacci number, except , which has no numbered chord above it. Our Lemma does not apply to . To …nish our cases, let's count the number of chord diagrams with only chord isolated. Then must contain the endpoints at arcs 1 and at 2, (Figure 2.) The options are at 3 or 4 or both, and, 4 or 5 or both, and... so on up to 3 or 2 or both; in other words, we have two less chords than usual to get crossed. Luckily, we're now experts at counting this: $_2$ We can use these results to count the number of null chord diagrams out of the 2 possible one-piece diagrams.

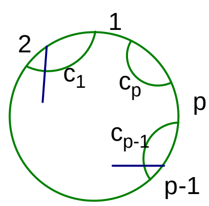

Figure 2. .

The above proof is longer than necessary for those readers familiar with the various ways the Fibonacci numbers occur in counting problems. We were unfamiliar with this Fibonacci structure and we were amazed to see our problem's approximate structure rewritten on Fibonacci puzzle pages. Our Fibonacci reference [3] gives as close a version as anybody else's.

So, the number of null, one-piece diagrams with chords is

$$ {}_2 + {}^{1}_{=1} \; 2 \; {}^{1} \; . $$

After calculating some values for this number, we noticed that it grows almost as fast as 2 . We programmed our calculators to …nd the ratio between

$$\sum_{=1}^{1} 2 \quad ^{1} \qquad \sum_{=1}^{1} 2 \quad ^{1} + \quad _{2} \quad 2$$

There is a Fibonacci formula using the Golden Ratio $= \frac{1+\sqrt{5}}{2}$. It states $= \frac{(-\frac{1}{})}{\sqrt{5}}$ [4], which turns the left-hand sum into a difference of two geometric series, when we divide by 2 Taking the limit as $\to 1$ squeezes the ratio of null chord diagrams to 1. The interesting math happens in the left-hand limit. Here, we substitute for and rationalize denominators, after taking the limit.

$$\lim \frac{1}{2^{p}} \sum_{=1}^{1} 2 \quad ^{1} = \lim \frac{1}{2\sqrt{5}} \sum_{=1}^{1} \frac{(-\frac{1}{})}{2} =$$

$$\frac{1}{2\sqrt{5}} \left(\frac{}{2} + \frac{1}{2 \quad + 1} \right) = \frac{1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{3 \quad \sqrt{5}} + \frac{1}{2+\sqrt{5}} \right) =$$

$$\frac{1}{2\sqrt{5}} \left(\frac{4\sqrt{5}+8}{4} + \frac{2 \quad \sqrt{5}}{1} \right) = \frac{1}{2\sqrt{5}} \left(\sqrt{5} + 2 + \sqrt{5} \quad 2 \right) = 1 \qquad \square$$

4. **Knot theory context.** We would like to take a moment to show how this paper started with knot theory. A knot is a loop in three-dimensional space, like a shoelace with the ends sealed together. The loop may weave in and around itself in all sorts of complicated ways. Telling knots apart has been, and remains, an important part of knot theory. A generalization of knots, called singular knots, has turned out to be quite useful in sorting knots. A singular knot is a knot which is allowed to pass through itself. A place where the singular knot intersects itself is a singularity.

A chord diagram represents a family of singular knots where each singularity is a chord. If we treat the knot as an oriented loop, we can draw a chord for the place where two points of the loop are treated as the same point, hence our naming the two endpoints of a chord with the same name. An isolated chord is like a pinch in a singular knot. Such a pinch is the least interesting of singularities and, in some calculations, the pinch causes the singular knot to contribute nothing. The chord diagrams with isolated chords get attention because they represent singular knots with pinches. This is also the reason why they are called null diagrams.

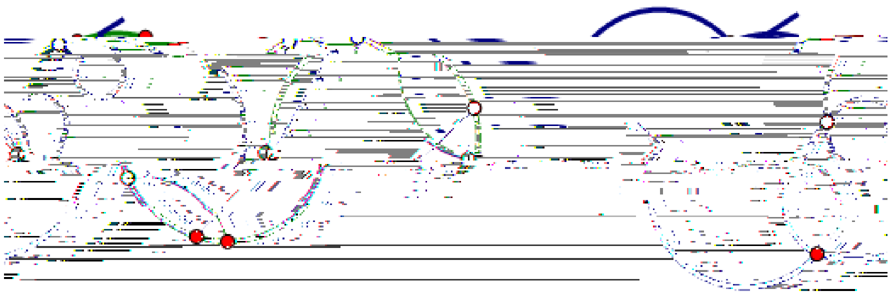

Figure 3. A chord diagram represents a singular knot.

In Figure 3, we show a singular knot which our chord diagram from Figure 1 represents, with labels suppressed. The arrow marker on the circle designates the starting point for drawing the singular knot. The two little blank sections on the singular knot indicate crossings, like overpasses on a road map. It turns out the crossings can be chosen in any way which does not change the relative order of the singularities. Traveling the two pictures will show that the isolated chord corresponds to the pinched singularity while the crossed chords require a more complicated arrangement in the singular knot.

There is a signed sum of all our one-piece diagrams with chords which has an abbreviation as a single symbol, called a Jacobi diagram. Our research began with Jacobi diagrams, some of which led to the restricted diagram structure in this article. The references [1] and [2] show the details for Section 4. The interested reader should consult them for more information on the knot theoretical context.

ACKNOWLEDGEMENT

The Marguerite and Dennison Mohler Foundation funded a summer research project in knot theory at Aquinas College. I appreciate their support. I have to thank the referees for their thoroughness and superb advice.

REFERENCES

[1] D. Bar-Natan, On the Vassiliev knot invariants, Topology 34 (1995) 423 - 472.

[2] S. Chmutov, S. Duzhin, and J. Mostovoy, CDBook Introduction to Vassiliev knot invariants, draft of book, Online at http://www.pdmi.ras.ru/~duzhin/papers

[3] A. Tucker, Applied Combinatorics, (John Wiley & Sons, 2002)
[4] D. Cohen, Basic Techniques of Combinatorial Theory, (John Wiley & Sons, 1978.)

Department of Mathematics, Aquinas College, Grand Rapids, MI 49506.
kraemjan@aquinas.edu, mcdanmic@aquinas.edu

Jane made it through Abstract Algebra and this research, despite her muscular dystrophy. She has presented her work at Michigan State and Aquinas College's Taste Of Undergraduate Research. She enjoys a good book, lunch breaks outside and joking around with the other math students.